Numerical Identity, Algorithmic Operationality and Self-Referential Paradoxes

Boris D. Grozdanoff

Philosophy of Science Department, IPhS

Bulgarian Academy of Sciences | DISI | QAISEC

grozdanoff@gmail.com

Abstract: The problem of numerical identity is closely related to identity of objects and self-identity, as formalized in the logical Law of identity. Here I propose a novel approach to analyzing numerical identity within the general framework of the concepts of operatio and operandum, which, as I argue, are to be found at the foundations of mathematics, logic and natural language. The approach introduces the notion of algorithmic history of sentences, both formal and in natural language, where the emergence of a well formed sentence develops on strict stages, reflected by the steps of standard algorithms. This algorithmic approach allows for two novel results: first, it permits to rigorously discriminate computationally accessible structure for sentences in natural and formal languages, which makes them rigorously available for computational operations and programs development. Second, it presents a fresh look at classical self-reference paradoxes as the Liar's paradox and Russell's paradox, where, as I argue, the paradox is only apparent since it cannot form on a single step of the algorithm of any sentence S, since any of S's interpretations and self-references develops, necessarily, on a different step of the algorithm.

Keywords: identity, numerical identity, algorithm, metaphysics, operandum.

In both logic and mathematics, when in a search for common elements of isomorphism, we see that there is perhaps no other promising candidate than the fundamental distinction between an *operatio*[1] and on *operandum*[2]. We see it in the earliest historical examples in Euclid's *Elements* where elements of geometry are defined explicitly as non-actions (points, lines, angles, etc.), but which are the only objects that allow for (geometrical, in this case) actions to be performed at all, such as construction (of a point), of a circle, translation (of a line), reflection (of a triangle, through a line), rotation (about a point) and

[1] *Operatio*, *operationis* (f.) is a Latin noun that originally denotes an activity, an action, particularly an action of a natural origin, like in "A lightning strikes the field". Its semantic essence, preserved from Latin into upcoming roman and roman-influenced languages (such as English) denotes an action.

[2] *Operandum* is the neutrum form of a participle of the verb *operō*, meaning "that which is to be worked".

others. The same most general structure we see in (natural) numbers, where the distinction between a number and the first and most fundamental operations over them, addition and subtraction, is the first distinction we see in the elements of, “elementary” arithmetic. Thus, any natural number is different from any arithmetic operation and any arithmetic operation can only be performed over numbers (like in expressions of the kind of 5 + 7) or (ordered) sets of numbers (like in expressions such as (5+7) + (4+7)). This difference is highly non-trivial and is best captured in the terms of operandum and operatio. Initially, before the proliferation of the current multitude of mathematical domains, the fundamental elements of geometry were points and geometrical constructions and motions allow for the emergence of lines, figures and the like. The fundamental elements of arithmetic were natural numbers and the arithmetical operations of addition and subtraction allowed for reaching the non-trivial results of arithmetics. The points in geometry and the natural numbers in arithmetic serve the function of *operandae*, i.e. the things or objects that are operated on. The constructions and motions in geometry and the arithmetical operations in arithmetic serve the function of *operatii*, that is, of actions over the respective *operandae*.

If we redirect our observation from the domains of mathematics towards the domains of natural language and logic we encounter the same general structure. In Indo-European languages[3] we see that the fundamental and first elements of the language, which cannot be reduced to other grammatical objects, are nouns (including names) and verbs. All other forms are reducible to them and serve them, via complementation or modification. Generally, nouns express *objects* in any natural language and are not expressing an *action* even if further evolved forms might express an activity as a noun or a noun like grammatical form (like in “rain” or “fall” in English). Verbs express actions. Neither nouns nor verbs can serve its opposite function on the same level and not on a meta-linguistic level and none of them can be derived from or reduced to the other in its function.

Thus, we seem to have a justification to be interested in the distinction between an operandum and operatio as a founding functional distinction that is found in natural

[3] For our current purposes I will ignore the extremely interesting but very different case of hieroglyphic or pictorial based languages, such as Ancient Egyptian, Chinese, Japanese and many others, for an operational analysis that aims to cover them would need to account for their pictorial foundation and its connection to semantics and communication as common functions of all natural languages. Such analysis is certainly of great interest and much needed but is currently beyond the scope of this paper.

languages and in formal systems. It is therefore of further interest to investigate its possible role in problems of identity and in particular, in the problem of numerical identity. In what follows I will regard the problem of numerical identity solely as a metaphysical and not as an epistemological problem, as it figures, in my understanding, in the famous *Identity of Indiscernibles principle* by Leibnitz.[4] Even if usually taken as an ontological and thus as a metaphysical principle, the key function of the concept of discernibility is of obvious epistemic nature since it both comes to and refers to the epistemic action of discerning, which is a function of the epistemic faculties of humans. Even if instructive about an ontological status of an object it is in essence an *epistemic* argument establishing this status, rather than a metaphysical vs. epistemic state proper. Therefore, and due to the metaphysical interest having priority for my current purposes here, I will exclude epistemic proper criteria for identity in the pursuit of arriving at a sensible understanding of numerical identity.

In an *analytic expositio* of the expression "numerical identity" we observe two key notions and a relation that orders them in it: the notion of number and its derivative *numerical* and the notion of identity. The relation between the two in the expression is quite clear and, on the one hand, it specifies the identity in the expression as being of a certain specific kind, the numerical kind; whereas, on the other, introduces the predicate «numerical» as designating a standard metaphysical property of x being numerical, which is most natural to be interpreted semantically as x being capable of having a number. For the purpose of brevity and simplicity we will ignore here all numbers but the natural numbers and we will interpret "x is numerically identical to y" where it could be both that y = x and y≠x.

Traditionally, the identity operator refers to a relation in logical formalism but also, identity is clearly not an entity of the operandum kind but rather an entity of the operatio kind. It is not a non-action x but it is open for both an interpretation as a metaphysical property (as in "Water is wet", W_w) and a metaphysical relation (as in "John loves Anne", jLa). Since, however, in the case where identity is accepted as a property and we decide that it is better to designate it with a predicate, this predicate would be a two place predicate since anything identical, x, would only be capable to be identical if and only if it is identical to something other than itself, even if this other is *another instance of the same x*. The meaning of the expression "x is identical" is clearly

[4] Leibniz formulates it in the *Discourse on Metaphysics*, Section 9 (Loemker 1969: 308).

incomplete without complementing it with another object, be it x or y and between the two expressions "x is identical" and "x is identical to y" we see the latter as having a completeness in meaning unlike the former. The semantic function in the two-place predicate formalism of the identity concept reveals the relational and thus – the *operational nature* of its proper semantic interpretation: it only acquires its meaning if it connects an x and a y. This, however, is exactly what relations do even if a property emerges out of relational expressions. In our inquiry whether identity is relation proper or a predicate proper we see that the former is neither derivable nor reducible to the latter whereas the opposite is quite conceivable. The interpretation of identity as a relation is also simpler than the interpretation as a predicate. And, in the frame of our operandum – operatio distinction (OOD) relations are clearly *operatii*.

In the light of the OOD an x could be numerically identical to itself or numerically identical to some y that is not x. The second scenario is more obvious as to the relational nature of the identity in its numerical kind. If x is an orange and y is an apple x is numerically identical to y since numerically the orange has a numerical value of 1 and the apple has the same numerical value of 1. If it were any other case where, say, the numerical value of y is any other value but 1 x would not have been numerically identical to y. Trivially, whatever the numerical value of x, if x is to be examined with respect to its numerical identity to itself, x would necessarily be numerically identical to itself. Non-trivially, if we examine the question if x having an identity at all has some relation with x being numerically identical to itself, we immediately observe that any conceivable fact of an emerging metaphysical individuation of any entity as a metaphysically proper object x involves the simultaneous emergence of x's numerical identity. For no individuated object can fail to possess a quantity, that is, a numerical value and thus to emerge as an object also in virtue of being numerically identical to itself. In the conceivable but very vague and hypothetical case where an object can emerge with no definitive numerical value one might argue that in virtue of the none-presence of such value x might not be numerically identical to itself since no numerical identity can take place if no definite numerical values are available for the both sides of the relation. In this case, however, we can also argue that even if the numerical value is undefined this does not in any way mean that it is *differently undefined* in x and in y, or that it somehow changes due to its being non-definite. The property of being numerically non-definite does not imply a change of this non-definite numerical value since only a definite value can change.

Another interesting scenario is a quantum type superposition of values: say our x is in a state of a numerical superposition of values where its Nv is a distributed cloud of possible values x can collapse to once observed, to follow the quantum framework of formulation. But if we are to follow the quantum analogy properly we would need to recognize that the set of all possible values has a probabilistic non-null distribution for any value in the cloud designating the probability for any of them to be the value x collapses to upon observation. Thus, there is an amount of definitiveness even to this exotic as it is, when considered in pure mathematics, scenario of numerical superposition. This analogy has other serious difficulties as well, since the quantum superposition is spatial and temporal whereas numbers in pure mathematics are abstract, causally inert and outside space, time and space-time.

The operational distinction allows us to discriminate fine structure in the notion of numerical identity that was perhaps not immediately available without it. Especially interesting is the numerical self-identity where x is numerically identical to x. While in numerical identity between x and y where y≠x we clearly observe that the *operandum* – *operatio* structure of the identity relation follows a short argument that consists of elements, steps, operations and transitions between steps. Characteristic of this argument is its clear *algorithmic structure*: it has a finite number of distinct steps with a definite order of sequence:

1. x (introduction)
2. y (introduction)
3. $=_n$ (introduction of numerical identity)
4. numerical value of x = 1
5. numerical value of y = 1
6. 1 = 1,
   (1 - 6), *therefore*
7. x $=_n$ y

In this setting the numerical identity is a relation that establishes a fact that holds between *two* metaphysically distinct objects, x and y, even if x is numerically identical to y, y≠x. Whereas in numerical self-identity (a) x is numerically identical to x *and* (b) x = x. From a metaphysical point of view it is very interesting what is the nature of the relation between the facts described by a and b.

For example, one question of significant metaphysical interest is the question does *self-identity imply numerical identity*? Another question of significant interest is *whether numerical identity is sufficient for self-identity*?

In the OOD framework of analysis we immediately observe that the answer to the first question is affirmative: once an object x is individuated as a metaphysical object it comes on board with its numerical identity; again, alternatives, as discussed above are at best inconclusive and much convincing as non-viable. The answer of the second question, however, is far from immediate. And especially so in the OOD context. For if we accept that numerical identity is sufficient for self identity our argument would take somewhat of the following form:

1. x (introduction)
2. $=_n$ (introduction of numerical identity)
3. numerical value of x = 1
4. 1 = 1,
   (1 - 4), *therefore*
5. $x =_n x$

What we see here, as very distinct from our first version of the argument about $x =_n y$, is that while in the x and y version we had one instance of x and one instance of y in the $x =_n x$ we have, in line (5) *two distinct instances of x*, one on the left hand side of the identity expression $x =_n x$ and one on the right hand side. Thus, it is non-trivial to ask whether the expression in (5) establishes a numerical identity between x and itself or *whether establishes a numerical identity between two instances of x*? For self-identity is by definition an identity of x with respect to itself and *not to an instance of itself*, expressed by the second instance of x at the right hand side of line (5). At best (5) can establish numerical identity between two, and potentially, infinite number of instances of x. Surely, the possible thesis that actual instantiation is necessary for self-identity is difficult to accept: we can conceive of an individuated x that is identical to itself, such as the natural number 1 and yet we need no demonstration of an instance of 1 in order to prove that it is self-identical. Thus, we see that *numerical identity is not just insufficient for self-identity but it is also not necessary for it since it is not achievable on one and the same step of the algorithmic history of S*. And self-identity cannot be developed in steps in order to be achieved: rather, self identity is assumed as a fact, best described by an atomic state, a state that simply persists and does not need a step-like evolution, much less transforming into self-instances in order to establish itself as a mere atomic state of self-identity as the one encoded in the logical Law of identity.

This reasoning suggests that the notions of numerical identity between distinct objects, the self-identity of a single object and the numerical identity of a single object have an operational nature and can only meaningfully emerge after a cognitive agent reasons via natural and formal language about them. The main argument that supports this thesis is the very operational nature of the above schemes of identity; but all those operations are not metaphysical operations that occur between metaphysical objects as they are by themselves but only within our linguistic rational apprehension of them. *The nature of numerical identity, both between distinct objects and in a single object, is operational and therefore symbolic and semantic yet not metaphysical*. These three distinctions are central in one of the most influential takes on the subject, the one put forward by Frege.

FREGE'S THREE TYPES OF IDENTITY

The OOD insight in numerical self-identity can be useful in the direction Frege explored when discussing identity and, famously, in *On Sense and Reference*. There he defines identity as

> "Identity gives rise to challenging questions which are not altogether easy to answer. Is it a relation? A relation between objects, or between names or signs of objects? In my *Begrifsschrift** I assumed the latter. The reasons which seem to favor this are the following: *a=a* and *a=b* are obviously statements of differing cognitive value; *a=a* holds a priori and, according to Kant, is to be labeled analytic, while statements of the form *a=b* often contain very valuable extensions of our knowledge and cannot always be established a priori. The discovery that the rising sun is not new every morning, but always the same, was of very great consequence to astronomy. Even today the identification of a small planet or a comet is not always a matter of course. Now if we were to regard identity as a relation between that which the names *"a"* and *"b"* designate, it would seem that *a=b* could not differ from *a=a* (i.e., provided *a=b* is true). *A* relation would thereby be expressed of a thing to itself, and indeed one in which each thing stands to itself but to no other thing ..."[5].

The three options Frege discussed here are *metaphysical*, where identity is a relation between objects, *semantic*, where identity holds between names and *formal or syntactic*, where identity holds between signs of objects. All three options are illustrated by using the expressions of *a = a* and *a = b*.

Here I would like to shed some light on the illustration and discuss Frege's stance on identity. In the a = a expression it is *assumed* and not proved that the symbol *a* is the same symbol when it stands on the left hand side of the equation and when it stands on the right handside of it. Even if we agree that it is the same symbol, we should also agree that in the equation there are *two instances of it*. Thus, we observe that the instances of, allegedly or arguably, the same symbol, are present in the illustration of all three options of identity Frege presents us with. An instance, by definition, could only be, however,

[5] Frege, Gottlob (1892) "On sense and reference" as reprinted in A.W. Moore (ed.) *Meaning and Reference*. Oxford: Oxford University Press, p. 23.

another copy of one and the same thing. And if the thing were not one and the same, like in the a=b case we cannot call it an instance. For *b* is not and could not be an instance of *a*, when we regard it on the level of symbols, and not, at the operationally deeper level of reference or of the level of sense, as it is within Frege's theory of sense and reference. Thus, we see a reason to regard a seemingly same symbol, a, same as a, as an instance of the symbol a. The alternate conceivability of the left *a* being an instance of *a* and the right *a* being an instance of *b* is untenable, due to the obvious graphic difference between *a* and *b*.

A different option is also interesting. Is it not possible that both *a*s are instances of instances? And not of one and the same, hypothetically first, *a* symbol? This conceivability seems much closer to the status of a genuine possibility for we see no reason how this scenario is rationally excluded. In it, however, we need to begin to track «back» the instances' inheritance paths of both *a*s and again, we see no obvious reason to end it at a certain place rather than another one. Yet, logically, if we accept that any *a*, the left and the right, are instances, which we seem to have much better reason to do than not to, we should also agree that instance is intentional and thus it is impossible there to be an instance of *x* where *x* does not exists as individuated entity. Thus, we can accept that the *x* before every instance of *x* exists and if we decide to ignore loops of instances we need to trace back each instance to its original *x*. And we can certainly distinguish, as non trivial, between *x as an instance of* is not the same as *x which is not an instance of*. This latter *x* is the original, to assume, *x*, which by itself is not an instance of the same *x*, since it is the same *x* and as such it cannot be an instance of itself: $x \neq x_{ix}$.

THE IDENTITY OF NUMBERS

The approaches to discuss the identity, as Frege frames it influentially in *On Sense and Reference*, can be generalized as linguistic or symbolic, where identity is considered as a relation or an alternative category, which holds between signs of objects, that is, between symbols and ordered strings of symbols. Or, it could be semantic, as a relation between objects, where objects are taken as meaningful strings of symbols. In following these two lines we should not, of course, restrain ourselves in the frame of Frege's own sense and reference distinction. But we can certainly use the productive and non-trivial distinction between symbols and their references, the alleged objects they stand for; in case they do. Perhaps there is no better place to begin with than numbers and we can consider the natural numbers. Thus, let us take the natural number “1” with respect to the question of its identity. It is obvious that we can relate 1 and identity in a multitude of ways. We can ask, meaningfully, whether we can attribute the category of identity with respect to 1 as it is considered by itself; or, as it is considered with respect to a different symbol; or with respect to its eventual reference; or with respect to the eventual reference of the other symbol(s). As we consider the applicability of identity to 1 as it is by itself, we can inquire if 1 does have an identity, implicit or present in another way. The immediate answer to this is that 1 certainly does have an identity, and in more than one way, since it is individuated as 1 and not as something else and on the bases of its very presence 1 does come and does emerge as what it is, 1, by virtue of its being individuated with respect to all natural numbers that are not the same as 1. This way of discovering the identity of 1 is to a great extent a contrasting kind of identity, which is formed in virtue of its *difference to something* and all that is not itself. It is interesting to note that the contrast basis does not need to be individuated in order to become able to serve as a contrast individuation basis for the emergence of the individuated 1. That is, the non-1 that we discovered clearly as a contrast basis for the individuation of 1, does not need to have its own identity in order to serve as a contrast basis for 1's identity. 1 might be considered individuated with respect to being formulated and introduced, first and second, as *different from*, say, 0. But it needs not be. Thus, we discover that identity requires the notions of difference for its distinct emergence and the notion of some individuating property which serves as a substance of the individuated difference, the semantic content. We can discover a multitude of properties in 1: symbolic, semantic, referential, applicative and many others. Thus, we arrive at the notion of property, which is the main constitutive basis of traditional theories of objecthood in metaphysics, like the bundle theory[6] and the bare particular theories of identity.[7]

Can we, further, ask if 1 is identical to itself? That is, can we inquire about a possible identity attribution of 1 with respect to itself? It again becomes immediately clear that the very alternative, that 1 is not identical to itself, or otherwise, that 1 cannot be identical to itself, is absurd. In our considering 1 we find that in our very being able to do so with no difficulties or obstacles of rational nature, we can only consider it because

[6] Locus classicus for the formulation of ontological bundle theory is Hume, in “A Treatise of Human Nature”, Book I, part IV, sec. 6.

[7] Modern reintroduction of the bare particular theory of objecthood is presented by Moreland, J.P. (1998) *Theories of Individuation: A Reconsideration of Bare Particulars*, in “Pacific Philosophical Quarterly”, 79, 251–263.

1 is individuated as 1 and thus it does have an identity and because of its having an identity it is capable of being identical. Ans, as we saw above, because it is identical, there must be something that it is identical to, since identity is a category of (at least) two-place relation. And since so far our considered poor universe of consideration has but only two inhabitants, 1 and it's all non-one, in order to have and to preserve its identity 1 must by logical necessity be identical only to itself, that is, it can only be self-identical. For, again, if it were not, it would not only have failed to be identical to itself, but it would have also failed to have an identity, that is, to not be 1, which is not the case, it would have formed a contradiction and it is obviously not possible. The lack of self-identity would eliminate the metaphysical individuation of 1 and since 1 is as we observe it in its symbolic representation «1» this again cannot be the case. Therefore, we can justifiably accept that 1, as any other object of similar inquiry, is and cannot fail to be identical to itself, arriving heuristically at the famous law in history of logic and philosophy that everything is identical to itself. [8]

What is again notable is that in our discovery of 1 being identical to itself we have used identity as a relation category. Just as 1 is of the category of a *relatum* here, by being the thing that is being related, in this case - to itself. How can we correctly describe the operation of 1 being identical to itself? Can we regard it as an operation in our domain of description only, that is, the relatum and the relation are related only within our description of them? That would be an occurence of identity within Frege's syntactic domain of symbols. Or, is it an occurrence, a fact, that takes place on the level of the object of our syntactic description; namely, 1 as an object and *not as a name of an object* and its identity, to itself, as a relation and *not as a name of a relation*? It again becomes immediately clear that if it were the case that the identity of 1 would be within the domain of symbols and names we could accept it and also affirm that it does not simultaneously take place in the level of 1 as an object. Alternatively, we can deny it and accept that it does take place *only within the object domain*, or perhaps that it does take place on both domains, the symbolic and the object domain. A justification for the second scenario is us realizing that if our language is to say something true of a reality outside itself, that is, if we would prefer that our language is able to actually refer to objects outside the very domain of language and symbols, it must be also the case that identity takes place on the domain of objects as well.

[8] The Law of identity in Logic is first found in Plato's dialogue *Theaetetus* (185a); later it is made explicit by Aristotle in *Prior Analytics*, Book II, Part 22, 68a.

If so, however, we face two distinct kinds of what we initially assumed was only one identity: the symbolic identity between the symbols of 1 and itself and the object identity between the referent of 1 and itself. It is again, immediately clear that such two identities are at the same time distinct, in virtue of the first being a description identity that holds between 1 and 1 as symbols and in virtue of the second being an object or metaphysical identity that holds between the referent of 1 and the same object.

If 1 is identical to itself and there is 1 that is not just a name of 1 but an object, referred to successfully by the symbol 1, we can inquire about the instances of 1 as symbols and what is their stance with respect to identity. Thus, if we accept that there is an object, referred to by the symbol 1, and we call this object "number 1" we immediately see that we face a multitude of symbolic instances of 1, as in 1,1,1,1,1,1,1 ... etc. And each one of them appears referring to the same number we referred to so far by appearing as a single symbol of 1, but effectively instantiated multiple times. A question which arises is "Do all instances of the symbol "1" refer in the same way to the number 1?", but also the question "Are the instances identical between themselves?" and also, "In case they are identical between themselves, are they identical in the same way as the symbol 1 to itself and to no other instance or like the number 1 to itself?".

THE *OPERANDUM* AND *OPERATIO* DISTINCTION FOR PARADOXES

If numerical identity and the concluded role of instances for identity are operational and thus logical and linguistic of nature yet not metaphysical proper we can attempt to harness this result in order to see if it can be productive in classical logical and linguistic tasks that are usually one of the final success measures for some candidate theses. We can approach paradoxes in logic and language with the operational structure of numerical and instance-analyzed identity, such as the Liar paradox,[9] the Russell's[10] paradox and perhaps others.

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[9] The earliest formulations of the Liar's paradox can be found in Megarians, Aristotle and Cicero and it is often known also as *Epimenides' paradox*. For a modern discussion on its role in logic see Chihara, Charles (1979) "The semantic paradoxes: A diagnostic investigation", in *Philosophical Review*, 88(4): 590–618. doi:10.2307/2184846

[10] Russell's paradox reveals a contradiction in early Set theory and results from a loop interpretation of R, the set of all sets that are not members of themselves. Its formulations can be found in Russell, Bertrand, "Correspondence with Frege". In *Gottlob Frege Philosophical and Mathematical Correspondence*. Translated by Hans Kaal., University of Chicago Press, Chicago, 1980; and also in Russell, Bertrand (1996) *The Principles of Mathematics*. 2d. ed. Reprint, New York: W. W. Norton & Company; First published in 1903.

A sentence, natural or formal, acquires meaning and becomes first susceptible to be interpreted in truth and meaning only after its formation which is essentially an attribution of a predicate to a subject or an act of quantification; sentences in natural language, in simple declarative forms like “The Sun shines”, exhibit the same logical form, depending on the logic of choice. We can distinguish between (at least) two distinct stages in the initial sentence formation:

1. The introduced availability of the elements (formalized as subject, predicate or relation)
2. An attribution between them (i.e. – predicate to a subject, quantifier over a subject, etc.)

After these two distinct stages of every sentence’s history the possibility first emerges for a sentence to be well formed. Once a well-formed sentence is available we can interpret it with respect to its meaning and truth. But also, and operationally significantly, we are already presented with an operational or effectively, *an algorithmic structure of the history of the sentence*, which at every step records its elements introduction, its elements discharge, operations execution (predication, quantification or logical operators execution) and their sequence. This is a rigorous record that also allows us to observe the domain of interpretation of the sentence, for example when a sentence is applied to a certain semantic state of affairs. Or, when it is applied to other sentences. Or, most interestingly, *when it is applied to itself*, like in the general case of loop self-referential logical paradoxes.

In the light of our OO distinction we immediately see that in its algorithmic history any sentence, leading to a paradox or not, first emerges as well formed, and only then can be interpreted or applied to another sentence or to itself. In the case of classical paradox application of a sentence of the form “This sentence is false” we first need its well formation that allows at all its semantic interpretations, in meaning and in truth. The act of interpretation, essentially in reference of the Fregean type, of a sentence S, as referring to itself (for a sentence, of course, could serve as its own reference) can only occur in the algorithmic history of S at a different and later subsequent step. Therefore, as we see within our OOD, the occurrence of S for the stage of self-reference is effectively a *different instance of S* that *is not identical to the first instance of S*, at the stage of well-formed emergence. That is why every self-referential paradox collides two distinct instances of one self-identical sentence S. The instances, however, are not self-identical and each comes with its own, sentence-instance distinct semantic interpretation, in meaning, in truth and in reference or application. Thus, when we interpret the S in itself what we algorithmically do is to take the first instance of S, S_1, and to interpret it for a first time, I_1, in a second instance of S, S_2, and then to apply it to S but only towards the first instance of S_1, from the stance of the second instance of S_2.

On this reading no self-referential paradox can emerge at all on the level of the truly same sentence S, because at every stage of the algorithmic functioning of S the sentence figures *via its instances* and not in an abstract non-instantiated way. All instances of S, however, as we observed in the above sections, are *not identical to each other*. They are different instances of S and each comes with its own semantic interpretations and references and applications: those are also and even much more non-identical among themselves. The interpretation of S_1 is only an interpretation of its meaning. The content of this interpretation allows, at a different and later stage, the application of S to itself, but this can only happen, at earliest, by applying a second instance of S, S_2, to S_1. Significant is also that the S_1 as referred to by S_2 can only happen in a third instances of S, S_3, where S_3 is loop referring to S_1, but it still is a third instance of S, S_3 and $S_3 \neq S_2$, just as $S_1 \neq S_2$ and $S_3 \neq S_1$. Thus, the self-referential paradoxes cannot strictly emerge on the same sentence, S, because S does not figure in the algorithmic history as S proper, but only as separate and distinct instances of S. Those preserve the initial meaning of S and do not affect its identity, but figure with its own, non-identical to each other semantic interpretations. And the pairs [S_n + $S_{n \text{ semantic interpretation}}$] are not the same, they are stage different and not identical to themselves. The distinction between any two steps in an algorithm is non-trivial and especially so in self referential occurrences where an instance of S in one step is loop applied to another instance of S in another step of the algorithm. And, I argue, each sentence S has a sequential step history of emergence and evolution that has a finite number of steps and each step is sequentially distinct from every other. Therefore, self-referential paradoxes cannot emerge on a same step, which would be the most powerful and most rigorous occurence of a formal paradox. Instead, what we falsely accept as the same step paradox occurs in a loop of at least two or more steps. This adds internal semantic and algorithmic structure to the paradox and dissolves much if not all of its paradox power that stimulated the sense of rigor of so many great philosophers, logicians and mathematicians.

Here, I believe, we need to introduce a key distinction between any sentence or expression S, be it formal or in natural language, and its history. A single S can have a myriad of branches, each for every actual communicative or interpretative situation, and while the original S remains the same its role in every of these branches is different. How do the semantic branches contribute to the identity of S, to itself-identity? Does S comprise a complex multi-identity in the totality of its actual and possible semantic branches of different algorithmic histories? Or the self-identity of S excludes the branches? It is clear to me that any sentence S acquires portions of its identity via figuring in actual and possible semantic events and all of them reveal an original semantic aspect of S. This includes the linguistic and formal practise of language users which has always been a guiding light in the philosophical research on the problems of language. Thus, we cannot and should not ignore any usage of S in any algorithmic history where S is well-formed as S and the algorithms are well-defined. Thus, I argue that the true self-identity of any sentence S is built up by the ordered totality of all histories where S figures. Self-identity of includes the ordered set of all instances of S. To give a somewhat apt analogy: the identity of a certain concrete electron in general theory of relativity is not given by the electron itself, but by the worldline of the electron, which encompasses the history and the complexity of its whole existence. The same should hold for the identity of S, where S's instances, figuring in actual histories of usage, seem like a natural counterpart of all events-interactions of the electron.

CONCLUSION

No sentence is historically atomic, it forms and functions at well distinct stages. This is best grasped in a classical algorithmic structure which also brings the immense benefit of computability, both on the steps and on the elements of sentences that figure in those steps, via instances of sentences and distinct steps interpretations. We saw that it is productive to observe that any sentence S forms by the introduction and transformations of only two distinct elements of an atomic nature: an *operandum* and an *operatio*. To put it bluntly, an operandum is anything that figures in the sentence as a thing. An operatio is any action over it. If this structure begins to look familiar it perhaps it indeed is, for we know it from the interdisciplinary study of algorithms, ever since Al Horizmi formulated the first structure of an algorithm.[11] Algorithms have many benefits and are an essential structure of modern computers and their computations. The connection between sentences of natural language, expressions in logic and pure mathematics has always been

[11] For the history of algorithms see Berlinski, David (2001) *The Advent of the Algorithm: The 300-Year Journey from an Idea to the Computer*, Harvest Books.

elusive to philosophy. Rare and formidable exceptions are the likes of Gödel numbering.[12] They occur much more rarely than needed even if they effectively allow for the formation and exploitation of rigorously connected elements of formal and natural languages and mathematics. Such connections, here present in its general form of sequential algorithmic history of all sentences, formed formally or in a natural language, open the immense power of computability over them. Programs could be developed to prove syntactic and semantic facts for natural language and as well as formal facts and, like Gödel, formal meta-facts.

The simplest algorithm is an algorithm with two steps with one operandum and one operation. In the first step the operation is executed over the operandum (as in the predication of P to s) and in the second step we can minimally describe the novel state of affairs that obtained after the event in step 1. The atomic content of every algorithm is the execution of an operation over an operandum, as this can include the mere description of an operandum, like the introduction of a term in a logical argument, a term that allows, along with additional terms, the formal execution of logical operations like conjunction or implication. *The key feature of any structure of an algorithmic nature is that it exhibits isomorphism with the atomic structures and elementary operations in logic, mathematics and natural languages*. In predicate logic this is typically a predication over subject or a quantification over subject or predicate. In natural language it is the affirmation, declarative as in description the «Sky is blue», or even in order expressions like "Sit down", where even in an apparently minimal form of "sit" we see that the contents of the action required by the meaning of the order unpacks a sentence of the form "John should sit down in the chair". The other key feature of particular logical but also mathematical significance is the non-trivial distinction between the steps of the algorithm, which we can denote as $s \neq s_{n+1}$, where s is the first step of any algorithm and n+1 is the positive whole number denoting the number of the succeeding step.

[12] Gödel employs first the technique of the numbering that today carries his name in Gödel, Kurt (1934) "On Undecidable Propositions of Formal Mathematical Systems" (mimeographed lecture notes; taken by S. Kleene and J. Rosser), reprinted with corrections in Davis 1965, 41–81, and Gödel 1986, 346–371.

In a simple version of the Liar's paradox we can non-trivially distinguish between the stages of the sentence formation and the stages of the sentence application. We can further distinguish between syntactic stages and semantic stages. The syntactic stage of formation of the sentence ("What I am saying in this sentence is false") S ends with the syntactic well-formedness of S. Once syntactically well-formed S is available for semantic interpretations in truth and meaning. It also becomes available for an *execution*, if it contains a recipe for action, only after it has been interpreted successfully as a meaningful sentence and *the action instruction has been extracted from the meaning*. Note that truth interpretation in this scenario is not necessary at all.

The extraction of action instruction from a sentence' meaning represents the formation of a structure of steps and actions from the semantic structure of S, which consists of singular terms and a semantic order that holds them in S; and not, say, in S_1. The extraction is highly non-trivial for there might be numerous mistakes in it that would render the application of the sentence as a sequence of actions, inadequate to the semantic content of S. The unique identity of each step and operation in the execution of S (as applying it to S, for example) allows on the one hand, the very application of the extracted as an algorithm semantic content of S, and on the other, the very formation of the algorithm. To illustrate, the term "lie" as in S is a mere term, whereas the extracted as an instruction for action command "lie", as found in the extracted from S algorithm, is of a different nature; it is an executable instruction that denotes *an operation over a certain operandum* which, in self-referential loop cases, might be the very sentence S.

The self-identity of a sentence S is exhausted by the totality of all algorithmic histories of the usage of S' instances.

References

Berlinski, David. 2001. *The Advent of the Algorithm: The 300-Year Journey from an Idea to the Computer*, Harvest Books.

Chihara, Charles. 1979. "The semantic paradoxes: A diagnostic investigation". In: *Philosophical Review*, 88(4): 590–618. doi:10.2307/2184846.

Gödel, Kurt. 1934. "On Undecidable Propositions of Formal Mathematical Systems" (mimeographed lecture notes; taken by S. Kleene and J. Rosser), reprinted with corrections in Davis 1965.

Frege, Gottlob. 1892. "On sense and reference" as reprinted in A.W. Moore (ed.) *Meaning and Reference*. Oxford: Oxford University Press.

Russell, Bertrand. 1980. “Correspondence with Frege”. In: *Gottlob Frege Philosophical and Mathematical Correspondence*. Translated by Hans Kaal. Chicago: University of Chicago Press.

Russell, Bertrand. 1996. *The Principles of Mathematics*. 2d. ed. Reprint, New York: W. W. Norton & Company; First published in 1903.